**RSTDPT (Constant)**

**Constant (in time) Perturbation**

Now I’d like to establish some general results for G and S. These are most easily (I think) worked out using the Fourier space representation of G and S, introduced in the last file. So recall we have:



In the last file, we took the Fourier transform of G and found,



Now let’s define the T-matrix (off shell T-matrix)



Can also write this as T = V + VGV, where G is the full Green’s function. And in terms of T, we can write the GF as:



So there.

**General Properties of G(ω), T(ω), S(ω)**

So we saw that in previous file:



If we expand G in its eigenbasis |n>, including not only the bound states |b>, but also the unbound states, |k>, we get:



So,



And we see that G has poles at all of its eigenenergies, n. Note these are the energy eigenvalues and eigenstates of H, not H0. The b are negative and discretely spaced, while the k are positive and continuously spaced. So G has discretely spaced poles along the ω < 0 axis, and a continuum of poles, i.e., a branch cut, along the ω > 0 axis. [after doing the delta function problem, and getting Gkk´(ω), we see there are poles at ω = k too. Why? Maybe because |k> and |k> have partly a delta function overlap, which is strong enough overlap to pick out a pole?] What about T? Since T is given by V + VGV, we can see that T will have the same pole structure as G, since V is non-singular. Let’s actually work this out a little more explicitly. So,



And so, introducing a resolution of identity in terms of the eigenstates of ,



The sum over *a* includes both the bound and free states of H. So really it’d be a sum over the bound states and an integral over the free states. So we might write this as:



where we use the fact that scattering state energies are unchanged (the continuum of positive scattering energies just gets mapped to the same continuum of positve scattering energies), just the wavefunction changes. So again, Tni will have poles at the discrete bound states, and a branch cut starting at ω = k(min), which we’ll take to be 0, and extending out to ∞. Note while Gkk´(ω) had also a poles at ω = Ek, I don’t think Tkk´(ω) will, as its numerator is the overlap of |k> with the potential, not with the exact scattering state |k> with the potential, and this overlap should be non-singular. Now let’s get Sni(t-t0). We’ll start from G. So we have, taking the inverse Fourier transform of one of the equations relating G to T,



where the En,I are unperturbed energyies. And since,



we have:



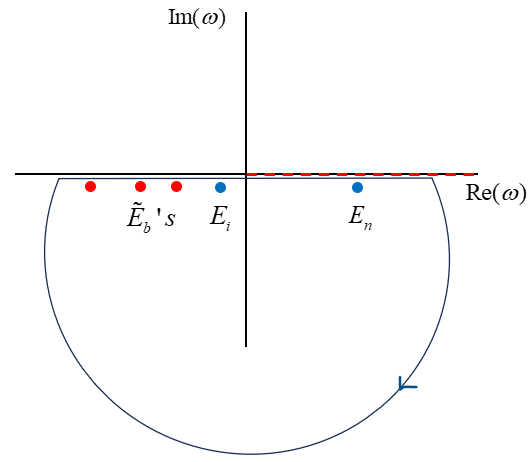
And next,

**Exact Results for Sn≠i(t,t0)**

Let’s use this information to examine the S-matrix for transitions between scattering states specifically. These are states |i> = |k>, |n> = |k´> which are momentum eigenstates. But we won’t really invoke the character of the states yet. So this will be completely general so far. Now we found,



And we can formally do the integral using contour integration in the complex plane. Since t – t0 > 0, we would have to close the integral down in the l.h.p., where Im(ω) < 0. The poles of the integrand are ω = Ei - iε, and ω = En - iε, as well as whatever poles might be present in Tni(ω). As aforementioned, Tni(ω) can be presumed to have discrete poles at positions ω = b – iε, where b stands for any bound state eigenenergies present in H. But Tni(ω) also has a continuum of poles / branch cut for all Re(ω) > 0. So our contour integration looks like this, where Ei and En are just whatever states right now, but will both eventually be scattering states, and so will eventually be along the Re(ω) > 0 line:



So doing the integral, using the Residue stuff from complex analysis, we’d have:



Now let’s add and subtract a term,



So we can write,



In the case where En and Ei are somewhat close, and Δt ~ 1/|En – Ei|, then the first term in the bracket makes the dominant contribution: the second should be much smaller as Tni(Ei) – Tni(En) ~ 0 one suspects, and the the third should be much smaller as it should average to zero considering the large oscillations in the exponent (but that’s only true if we can say that En– Ei << En – b. This won’t be true for a lot of systems, like spins for instance, whose unperturbed energies are uniformly spaced. This will really only work for Ei ≈ En, for scattering states really). So we can say,



And indeed, it would really work for all times larger than 1/|En – Ei| too, as indicated, provided the energy levels are close. This is what we found in the previous file, but now we do not have the implicit restriction that |ΔEni| > |Vni|, or that Δt ~ 1/|En – Ei| only. Now let’s specialize to off-diagonal transitions (n ≠ i). And we’ll look at the transition rate.



So we have:



Again, this is of course what we tentatively found in the previous file. But now see that it’s true for any potential strength, assuming Ei and En are the closest energy levels (by faaaaar), and that it also holds for all time on the order of and after 1/|En – Ei|. This formula says that the probability of transition from Ei --> En basically grows as 2(t-t0)|Tni|2 for t-t0 ≲ 1/(En – Ei), and when t-t0 ~ 1/(En – Ei), the transition rate will begin to descend back down and oscillate back and forth with amplitude |Tni|2/(En – Ei) and period ~ 2π/|En – Ei| for t-t0 ≳ 1/(En – Ei). So eventually, given discrete energy states, the transition rate will stop growing at time t on the order of the reciprocal of the energy difference, and begin to oscillate back and forth. But if we consider free/scattering states, which exist in a continuum of energies, then we can have En arbitrarily close to Ei. So given some initial state |i>, we can always find an |n> close enough to |i> to make the inequality |t-t0| ≲ 1/(En-Ei) true. And so as t progresses the transition rate of |i> -> |n> will grow with t, going as 2(t-t0)|Tni|2 for an ever narrowing strip of |n>’s centered about |i>, delineated by |En – Ei| < 1/|t-t0|, while transitions to those |n>’s outside this range will peel off and start oscillating. So in the long time limit, transitions to an ever narrowing strip of energies about Ei will continue to grow indefinitely towards infinity. In this sense, the transition rate behaves as a delta function. We can make this connection precisely by using the identity, limt🡪∞ sin(xt)/πx = δ(x) to write:



Note it’s *essential* that the E’s be part of a continuous spectrum to use this delta function identity. We’ll find these last few equations pretty well validated in the context of the time-dependent wavefunction for a delta function potential in a couple files from now.

**Exact Results for Gnn(ω)**

Now let’s look at the diagonal elements of the GF, or S-matrix. We should be able to extract information about the energy levels of the full H = H0 + V system, by examining the time-development of Gii. This is the likelihood that the state after evolution, will evolve into itself, i.e., will stay where it is. The likelihood that it does this is equal to the likelihood that |i> is an eigenstate of the new Hamiltonian. And so by examining its decay out of that state we can see how closely it resembles an eigenstate, and also the mixup of true eigenstates into it. We ought to expect the following structure from the FT:



where n are exact energies of the pertubed system, in contrast to the unperturbed energies En. And so we see that its poles are those of the system energies … and its renormalization factor tells us the resemblance of the state |i> to an eigenstate |ψn>. Now let’s postulate that this series can be resummed into the form:



Let’s expand the self-energy in orders of V,



And now let’s expand Gii(ω) in orders of V.



grouping by order we have,



And comparing to:



This brings us to the equations (implicit summation over repeated indices, except for i):



The first tells us that,



The second tells us that,



And the third tells us that,



So the self energy is:



Now observe that in the 2nd order term, the term subtracted off is the diagonal term. So the entire 2nd order term is just the principal value evaluation of the sum. In the 3rd order term, the diagonal elements are when j = k = i, and when j = i, and k = i separately. These three terms are subtracted off by the last two expressions. So at least up to 3rd order we may write,



We might recognize this series as the same as the Brillouin – Wigner Perturbation Theory formula for the energy corrections (and lifetimes). In fact I think it is the same. For discrete energies, we can take i0+ to 0 immediately, and so it does reduce to the Brillouin-Wigner formula. Mahan seems to confirm it is the same. Anyway, so the solution, going back to:



we see that the solution, ω = i, to the equation:



which in the present context is simply the pole of the GF, gives us the new energy and lifetime, τ, of the state, as argued in the BWPT file. i.e.,



Exact eigenstates would have infinite lifetimes, and so would have no imaginary part, i.e., they’d be real. To make our interpretation of the imaginary part (if present) of the pole more credible, let’s approximate Gii(ω) about this pole we ‘found’:



So we can write,



The last line, about Γ, follows, if we undo the Fourier transform. Then we’d have something like



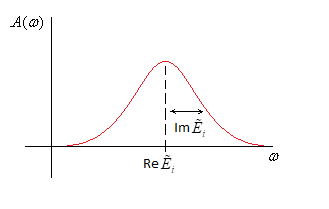
And can now see the imaginary part of gives the rate of decay (square G and we get basically the amplitude of the wavefunction squared, and then we get e-Γt term showing that the overlap of the state with itself, as time progresses, decays exponentially with rate Γ). Z would seem to be roughly proportional to how close our state |i> is to an exact state (higher the closer). We can look at this information in a different way. Borrowing from later discussion of GF’s, the spectral function can be defined as A = -(1/π)ImG. This would give us, formally,



So we get the density of states – basically the fraction of our state |i> that spreads into an exact state within energy range (ω, ω+dω). Can see that if we integrate w/r to ω. Then we just get 1. Taking our GF form above, constructing the spectral function from it, we have:



This is of form,



So we can see it’s centered about the real part of the pole of G, the new energy, and the width is related to the imaginary part of the pole, which is related to the scattering rate, the inverse lifetime. This makes sense because the larger the mix of exact energy states that goes into the our state |i>, the less of a true energy eigenstate it is, and the more quickly it should decay. Or, from the energy-time uncertainty principle, the greater the uncertainty of energy, the smaller its lifetime, and greater its decay rate. We can be even more explicit about the connection between the imaginary part of Σ and the lifetime when specializing to continuous energies, i.e., scattering states. Looking back at Σ,



I think we could then ignore the restriction on the summation, as the support for adding back in the j = i, k = i, etc. terms is zero in the continuum limit (but then we will need to keep the i0+ term). And then, we’d see that Σ(i,ω) is just the off-shell T-matrix, Tii(ω).



Can use this guy then, from Scattering folder (or see Appendix), which relates the off-shell T-matrix (guy on the left below) which we’ve been analyzing, to the on-shell T-matrix (guy on the right below), which we also discuss in the scattering folder:



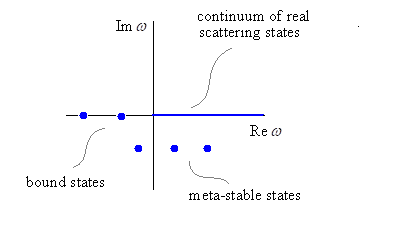
And note <k|T|k´> = Tkk´(ω=Ek). The poles of Σ(k,ω) should just be ω = k = Ek = k2/2m, since in the scattering case, the energies remain unchanged; just the wavefunctions change. So then, the scattering rate should be:



where we have used the familiar 1/(x+iδ) = P(1/x) – πiδ(x) identity to separate out the real and imaginary parts (this should only apply to free/scattering states; otherwise if discrete energy spectrum then can just take i0+ -> 0 and we get no imaginary part). And since k = Ek, we have:



just as we had found earlier, when examining the off-diagonal state transitions! So the GF has poles. We can plot these poles in the complex plane. True bound states will show up on the real axis, with no imaginary part. ‘resonant’ states will show up as poles in the lower-half complex plane – otherwise the lifetime would be negative. There should also be a continuum of positive energy (non-bound) states, just as we found for things like the δ potential, etc. A continuum of poles would amount to a branch cut.



and so we see Σii(ω) itself ought to have a complicated structure, with multiple poles and branch cuts. For that reason, we wouldn’t expect the perturbative series we developed above, which is basically analytic, to capture the full complexity of the ‘self-energy’, or in other words, to converge for all i’s.

**Appendix**

Maybe put that connection between Tkk(ω) and Tkk´ here? So recall,



And in terms of matrix elements, with implicit summation on repeated indices:



And let’s (re)introduce the on-shell scattering matrix, seen in adiabatic perturbation theory file, and also in the scattering perturbation theory files.



So now consider boxing our off-shell T matrix against two delta normalized k states <k|T(ω)|k>. We get:



Now insert resolution of identity in terms of the eigenstates of H, which we’ll call |>,



where k are the eigenenergies of H. And now revisiting the argument we made in the RSPT 3D scattering file (now we’re restricting |k> to be a scattering state; otherwise this scattering perturbative expansion shouldn’t really be valid, right? Right).



and here is the onshell scattering matrix and in the last line we used the general result from scattering perturbation theory, that,



So now we can say Tkk(ω) is equal to:



So can now explicitly see, again, that the poles of Tkk(ω) are the eigenenergies of the total Hamiltonian. Now I think that if the potential isn’t attractive enough to form bound states, then all the new energies should just be the old ones, so that while the wavefunctions may change, developing phase shifts and all, the energies remain the same (we saw this for instance when we looked at some of the repulsive potential scattering examples. So then we could say,

